

A SIMPLICIAL MODEL FOR PROPER HOMOTOPY TYPES

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ABSTRACT. The singular simplicial set $\text{Sing}(\mathfrak{X})$ of a space \mathfrak{X} completely captures its weak homotopy type. We introduce a category of *controlled sets*, yielding *simplicial controlled sets*, such that one can functorially produce a singular simplicial controlled set $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$ from a locally compact \mathfrak{X} . We then argue that this $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$ captures the (weak) *proper* homotopy type of \mathfrak{X} . Moreover, our techniques strictly generalize the classical simplicial situation: e.g., one obtains, in a unified way, singular homology with compact supports and (Borel–Moore) singular homology with locally finite supports, as well as the corresponding cohomologies.

1. INTRODUCTION

It is well known that if \mathfrak{X} is a topological space then its singular simplicial set $\text{Sing}(\mathfrak{X})$ (which, in dimension n , is just the set of continuous maps from the topological n -simplex to \mathfrak{X}) completely captures the weak homotopy type of \mathfrak{X} . The intuitive argument here is that $\text{Sing}(\mathfrak{X})$ contains all information about maps from spheres (of each dimension) to \mathfrak{X} since spheres can be triangulated, and that it also contains all information about homotopies of such maps since homotopies of spheres can be realized simplicially (i.e., $S^n \times [0, 1]$ can be triangulated). As such, homotopy theory can, if desired, be done in an entirely simplicial way.

On the other hand, $\text{Sing}(\mathfrak{X})$ contains no information about the proper (or locally compact) structure of \mathfrak{X} : the realization $|\text{Sing}(\mathfrak{X})|$ of $\text{Sing}(\mathfrak{X})$ is almost never locally compact, even if \mathfrak{X} is locally compact (or even compact). Thus if one wants to study the (weak) proper homotopy invariants (e.g., the Borel–Moore singular homology, various analytically obtained invariants) of a locally compact space \mathfrak{X} , it does not suffice to only consider the simplicial set $\text{Sing}(\mathfrak{X})$.

But of course one can develop, e.g., the locally finitely supported Borel–Moore singular homology of a locally compact \mathfrak{X} in a simplicial/singular way, by considering locally finite chains. (The original approach of Borel–Moore [1] is sheaf-theoretic in nature. The approach using locally finite chains is perhaps a bit folkloric, though it has been rigorously exposed in [2].) Information about whether or not a collection of singular simplices is ‘local finite’ is not contained in $\text{Sing}(\mathfrak{X})$; we solve this problem by replacing the category of sets with a category of *controlled sets* (or ‘sets with supports’), yielding *simplicial controlled sets*. In particular, for \mathfrak{X} locally compact, we naturally get a simplicial controlled set $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$. A feature of our approach is that it simultaneously generalizes the classical ‘compactly supported’ case and the proper case:

the singular simplicial set $\text{Sing}(\mathfrak{X})$ can be viewed as a simplicial controlled set $\text{CSing}(\text{MinCtl}(\mathfrak{X}))$ with ‘minimal control’ (see Remark 4.5), and doing so yields classical results.

After establishing basic definitions and properties, we endeavour to show that $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$ really does capture the weak proper homotopy type of a locally compact \mathfrak{X} . We do not do this in full generality, since doing so would require the development of a large amount of theory, in particular a corresponding category of ‘controlled topological spaces’ together with closed symmetric monoidal structures on these categories (see Remark 4.4), but only to the extent that is immediately possible (but we do give indications as to how to proceed in full generality). Just as $\text{Sing}(\mathfrak{X})$ contains all information about maps to \mathfrak{X} from finite simplicial complexes (a fact which can be expressed by an adjunction), we show that $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$ contains all information about *proper* maps to \mathfrak{X} from *locally finite* simplicial complexes.

Finally, we show how the locally finitely supported Borel–Moore singular homology (and the corresponding cohomology with compact supports) of \mathfrak{X} can be obtained easily and immediately from $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$. Indeed, the exact same process applied to $\text{CSing}(\text{MinCtl}(\mathfrak{X}))$ (which, as indicated above, corresponds to the standard singular simplicial set $\text{Sing}(\mathfrak{X})$) yields the usual singular homology with compact supports (and its corresponding cohomology).

We note that our terminology originates in coarse geometry; see in particular [6] (and also the author’s [3]). It goes without saying that there are also applications of our work to coarse geometry, even though we do not discuss them here. We have resisted changing our terminology of ‘controlled set’ to the arguably more transparent ‘sets with support’ (or ‘supported sets’), since the latter seems to lead to further unwieldy terminology.

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2. CONTROLLED SETS

Denote the category of (small) sets by **Set**. Recall that a set map $f: Y \rightarrow X$ is *proper* if $f^{-1}(\{x\})$ is finite for all $x \in X$. We have a subcategory **PSet** of **Set** with only proper maps, but it is rather poorly behaved. We will work with a better-behaved category which generalizes both **Set** and **PSet**.

A *controlled set* X is a set X equipped with a set $\text{Units}(X) \subseteq \wp(X)$ of subsets of X such that:

- (i) every finite subset K of X is in $\text{Units}(X)$;
- (ii) if K is in $\text{Units}(X)$, then so are all subsets of K ; and
- (iii) if K and K' are in $\text{Units}(X)$, then so is $K \cup K'$.

The sets in $\text{Units}(X)$ are called the *controlled subsets* of X and $\text{Units}(X)$ is called a *control structure* on the set X . A set map $f: Y \rightarrow X$ between controlled sets is a *controlled map* if:

- (i) for each L in $\text{Units}(Y)$, the image $f(L)$ is in $\text{Units}(X)$; and
- (ii) for each L in $\text{Units}(Y)$ and each x in X , $f^{-1}(\{x\}) \cap L$ is finite (i.e., the restriction $f|_L$ is proper).

One can check that (small) controlled sets and controlled maps form a category, which we denote by **CSet**.

There is an obvious forgetful functor $\text{Forget}: \mathbf{CSet} \rightarrow \mathbf{Set}$ which has a left adjoint $\text{MinCtl}: \mathbf{Set} \rightarrow \mathbf{CSet}$ defined putting, for any (small) set S , $\text{MinCtl}(S) := S$ with $\text{Units}(\text{MinCtl}(S))$ the set of all finite subsets of S . Not only does **Set** embed fully and faithfully in **CSet**, via the functor MinCtl , but so too does **PSet**, via a functor $\text{MaxCtl}: \mathbf{PSet} \rightarrow \mathbf{CSet}$: for any set S , put $\text{MaxCtl}(S) := S$ and $\text{Units}(\text{MaxCtl}(S)) := \wp(S)$.

3. SIMPLICIAL SETS AND TOPOLOGY

Denote the (topologists') simplicial category by Δ ; its objects are $0, 1, 2, \dots$ (labelled by geometric dimension) and the arrows $\mathbf{n} \rightarrow \mathbf{m}$ correspond to the weakly order-preserving maps from the total order on $n + 1$ elements to the total order on $m + 1$ elements. Recall also that for each n we have $n + 2$ *coface* maps

$$d^0, d^1, \dots, d^{n+1}: \mathbf{n} \rightarrow \mathbf{n} + 1$$

and $n + 1$ *codegeneracy* maps

$$s^0, s^1, \dots, s^n: \mathbf{n} + 1 \rightarrow \mathbf{n}$$

(we suppress the additional notation required to give the source/targets of these maps). Together with the *cosimplicial identities*, these maps generate Δ .

Recall that a *simplicial object* X in a category \mathbf{C} is just an object in the functor category $\mathbf{sC} := \mathbf{C}^{\Delta^{\text{op}}}$, i.e., a functor $X: \Delta^{\text{op}} \rightarrow \mathbf{C}$ (or a contravariant functor from Δ to \mathbf{C}). As usual, write $X_n := X(\mathbf{n})$, $d_i := X(d^i)$ (the *face* maps of X), and $s_j := X(s^j)$ (the *degeneracy* maps of X). Morphisms in \mathbf{sC} are just natural transformations, so a map $Y \rightarrow X$ between simplicial objects in \mathbf{C} is given by arrows $Y_n \rightarrow X_n$ (in \mathbf{C}), $n = 0, 1, 2, \dots$.

A *simplicial set* is just a simplicial object in **Set**. For each n , the *standard n -simplex* Δ^n is the simplicial set represented by \mathbf{n} , i.e., $\Delta^n := \text{Hom}_{\Delta}(-, \mathbf{n})$. The set of *n -simplices* of a simplicial set X is $X_n := X(\mathbf{n})$, or equivalently (by the Yoneda Lemma) $X_n \cong \text{Hom}_{\mathbf{sSet}}(\Delta^n, X) = \text{Nat}(\Delta^n, X)$, where as usual $\mathbf{sSet} := \mathbf{Set}^{\Delta^{\text{op}}}$; a *simplex* of X is an element $x \in X_n$ for some n (strictly speaking, one must keep track of the dimension explicitly since the X_n need not be disjoint).

The *realization functor* is a functor $|-|: \Delta \rightarrow \mathbf{Top}$, where **Top** is some 'convenient category' of topological spaces such as the category of compactly generated spaces (see, e.g., [4, Chapter 5]). On objects, it is given by

$$|\mathbf{n}| := \{(x_0, \dots, x_n) \in (\mathbb{R}_+)^{n+1} : x_0 + \dots + x_n = 1\} \subseteq \mathbb{R}^{n+1}.$$

If X is a simplicial set, we obtain its *realization* $|X|$ by the coend (in **Top**)

$$(3.1) \quad |X| := \int^n \mathbf{Top}(X(\mathbf{n})) \otimes |\mathbf{n}|$$

where $\mathbf{Top}: \mathbf{Set} \rightarrow \mathbf{Top}$ is the ‘discrete space’ functor and \otimes is just the ordinary cartesian product (since **Top** is cartesian closed). This can be made into a functor $|-|: \mathbf{sSet} \rightarrow \mathbf{Top}$ (again called the *realization functor*). There are a number of ways of describing the realization $|X|$ but they are all equivalent and all yield Milnor’s geometric realization [5].

Conversely, given a topological space \mathfrak{X} , we can form its *singular simplicial set*

$$\mathrm{Sing}(\mathfrak{X}) := \mathrm{Hom}_{\mathbf{Top}}(|-|, \mathfrak{X}).$$

It is clear that this yields a functor $\mathrm{Sing}: \mathbf{Top} \rightarrow \mathbf{sSet}$. The basic, but essential, result is that realization is left adjoint to Sing , i.e., for all spaces \mathfrak{X} and simplicial sets Y ,

$$(3.2) \quad \mathrm{Hom}_{\mathbf{Top}}(|Y|, \mathfrak{X}) \cong \mathrm{Hom}_{\mathbf{sSet}}(Y, \mathrm{Sing}(\mathfrak{X}))$$

(isomorphism of sets, natural in both \mathfrak{X} and Y).

Recall that a simplex of a simplicial set is *nondegenerate* if it is not in the image of any degeneracy map. A simplicial set X is *finite* if it only has finitely many nondegenerate simplices. If X is finite, then each X_n must be a finite set.

Remark 3.3. The definition of ‘finite simplicial set’ is a bit delicate: one cannot just demand that X only have finitely many simplices, since this can only hold if $X_n = \emptyset$ for all n . On the other hand, it is not enough to insist that each X_n be finite, since X could then still be infinite dimensional.

Since the realization $|X|$ is a CW-complex with one cell for each nondegenerate simplex [5, Theorem 1], the following is clear and well known.

Proposition 3.4. *If X is a finite simplicial set, then its realization $|X|$ is compact, Hausdorff.*

Simplices x, x' of a simplicial set X are *adjacent* if there exist arrows f, f' in Δ such that $X(f)(x) = X(f')(x')$. A simplicial set X is *locally finite* if each simplex of X is adjacent to only finitely many nondegenerate simplices. If X is locally finite, then all the maps with which X comes are proper (this follows easily from the injectivity of the degeneracy maps), i.e., X is actually a simplicial object in **PSet**.

Remark 3.5. It is not sufficient for our purposes to only insist that X be a simplicial object in **PSet** (but note that others, e.g., [2], define ‘locally finite simplicial set’ as such). Indeed, our definition essentially includes an assertion of ‘local finite dimensionality’. Our definition actually reduces to the a-priori-weaker requirement that each 0-simplex be adjacent to only finitely many nondegenerate simplices, since if two simplices are adjacent then they are both adjacent to some common 0-simplex. Observe also that if X is locally finite and finite in each dimension, then it is actually finite.

We have the following analogue of Proposition 3.4, which should also be clear.

Proposition 3.6. *If X is a locally finite simplicial set, then its realization $|X|$ is locally compact, Hausdorff.*

In fact, more is true. The category $\mathbf{sPSet} := \mathbf{PSet}^{\Delta^{\text{op}}}$ of simplicial objects in \mathbf{PSet} maps naturally to the category \mathbf{sSet} of simplicial sets, and we have already noted that locally finite simplicial sets ‘lift’ to \mathbf{sPSet} . Say that a morphism between locally finite simplicial sets is *proper* if it lifts to a morphism in \mathbf{sPSet} (i.e., if all the set maps involved are proper). We then get the following.

Proposition 3.7. *If $f: Y \rightarrow X$ is a proper morphism between locally finite simplicial sets, then the induced map $|f|: |Y| \rightarrow |X|$ on realizations is proper.*

Denote the category of locally compact, Hausdorff spaces and proper maps by \mathbf{PTop} (a subcategory of \mathbf{Top}). Combining Propositions 3.6 and 3.7, we get the following.

Theorem 3.8. *The usual realization functor yields a functor (which we continue to denote by $|-|$) from the category of locally finite simplicial sets and proper simplicial maps (which can be considered as a full subcategory of \mathbf{sPSet}) to \mathbf{PTop} .*

4. SIMPLICIAL CONTROLLED SETS AND TOPOLOGY

Our main goal is to generalize (3.2) to the controlled/‘proper’ setting. The essential difficulty in doing so is that $|\text{Sing}(\mathfrak{X})|$ is almost never locally compact, so the canonical maps $\varepsilon_{\mathfrak{X}}: |\text{Sing}(\mathfrak{X})| \rightarrow \mathfrak{X}$ (given by the counit of adjunction) in (3.2) are almost never proper. In this paper, we will only prove a special case of the controlled version of (3.2), since giving a general adjunction would take us too far afield.

Definition 4.1. *A simplicial controlled set is a simplicial object in \mathbf{CSet} (i.e., an object of $\mathbf{sCSet} := \mathbf{CSet}^{\Delta^{\text{op}}}$), and a morphism between simplicial controlled sets is just a morphism in \mathbf{sCSet} (i.e., a natural transformation).*

Define a functor $\mathbf{CSing} \circ \mathbf{MaxCtl}: \mathbf{PTop} \rightarrow \mathbf{sCSet}$ (our notation is justified in Remark 4.4) as follows. The forgetful functor $\mathbf{Forget}: \mathbf{CSet} \rightarrow \mathbf{Set}$ naturally yields a functor $\mathbf{Forget}_{\#}(\mathbf{CSing} \circ \mathbf{MaxCtl}): \mathbf{PTop} \rightarrow \mathbf{sSet}$, which we insist should coincide with our original functor \mathbf{Sing} (composed with the inclusion $\mathbf{PTop} \hookrightarrow \mathbf{Top}$). That is, if \mathfrak{X} is a locally compact, the sets underlying the simplicial controlled set $\mathbf{CSing}(\mathbf{MaxCtl}(\mathfrak{X}))$ are just the sets of the simplicial set $\mathbf{Sing}(\mathfrak{X})$, and similarly for the underlying set maps. We must also give the control structures $\mathbf{Units}(\mathbf{CSing}(\mathbf{MaxCtl}(\mathfrak{X}))_n)$: say that $S \subseteq \mathbf{CSing}(\mathbf{MaxCtl}(\mathfrak{X}))_n = \mathbf{Sing}(\mathfrak{X})_n$ is a controlled subset if it is *locally finite* in the sense that each point of \mathfrak{X} has a neighbourhood which meets (the images of) only finitely many simplices in S . It is easy to check that this does indeed define a functor (in particular, that all maps are controlled).

For any simplicial set X , $\mathbf{MinCtl} \circ X$ is a simplicial controlled set. (Our compositional notation $\mathbf{MinCtl} \circ X$ is due to the fact that X is a functor. Concretely, $(\mathbf{MinCtl} \circ X)_n = \mathbf{MinCtl}(X_n)$.) This embeds the category of simplicial sets fully and faithfully in the category of simplicial controlled sets. By considering locally

finite simplicial sets to be simplicial objects in **PSet**, we can likewise embed the category of locally finite simplicial sets and proper morphisms in **sCSet**: send each locally finite simplicial set X to the simplicial controlled set $\text{MaxCtl} \circ X$.

Our main theorem is the following.

Theorem 4.2. *For each locally compact, Hausdorff space \mathfrak{X} and each locally finite simplicial set Y , we have an isomorphism*

$$(4.3) \quad \text{Hom}_{\mathbf{PTop}}(|Y|, \mathfrak{X}) \cong \text{Hom}_{\mathbf{sCSet}}(\text{MaxCtl} \circ Y, \text{CSing}(\text{MaxCtl}(\mathfrak{X})))$$

(where, as before, **PTop** is the category of locally compact, Hausdorff spaces and proper maps). These isomorphisms are natural in \mathfrak{X} (and proper maps) and Y (and proper maps), and the isomorphisms are compatible with the classical adjunction (3.2).

More precisely, ‘compatible’ means that the square

$$\begin{array}{ccc} \text{Hom}_{\mathbf{PTop}}(|Y|, \mathfrak{X}) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{sCSet}}(\text{MaxCtl} \circ Y, \text{CSing}(\text{MaxCtl}(\mathfrak{X}))) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathbf{Top}}(|Y|, \mathfrak{X}) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{sSet}}(Y, \text{Sing}(\mathfrak{X})) \end{array}$$

commutes (above, we omitted some inclusions of categories).

Remark 4.4. The isomorphisms (4.3) do not constitute an adjunction since $\text{CSing} \circ \text{MaxCtl}$ does not produce locally finite simplicial sets. The complete, correct way to proceed is as follows:

- (i) Make **CSet** into a closed (symmetric monoidal) category, i.e., equip it with a ‘tensor product’ operation \otimes (in the obvious way) to make it into a symmetric monoidal category and then construct a compatible internal Hom.
- (ii) Define a category **CTop** of *controlled topological spaces* along with a ‘forgetful’ functor $\text{Forget}: \mathbf{CTop} \rightarrow \mathbf{CSet}$, and make it into a closed category in a way compatible with the structures on **CSet**. Essentially, the objects of **CTop** are spaces equipped with a suitable family of *controlled subspaces* (or ‘supports’) and the maps are continuous and proper relative to the controlled subspaces.
- (iii) Define various functors compatible with the existing ones:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{Top}} & \mathbf{Top} \\ \text{MinCtl} \downarrow & & \downarrow \text{MinCtl} \\ \mathbf{CSet} & \xrightarrow{\text{CTop}} & \mathbf{CTop} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{PSet} & \xrightarrow{\text{PTop}} & \mathbf{PTop} \\ \text{MaxCtl} \downarrow & & \downarrow \text{MaxCtl} \\ \mathbf{CSet} & \xrightarrow{\text{CTop}} & \mathbf{CTop} \end{array}$$

(where $\text{PTop}: \mathbf{PSet} \rightarrow \mathbf{PTop}$ is the obvious functor).

- (iv) Define $\text{CSing}: \mathbf{CTop} \rightarrow \mathbf{sCSet}$ by putting, for \mathfrak{X} a controlled space,

$$\text{CSing}(\mathfrak{X}) := \text{Forget}(\text{Hom}_{\mathbf{CTop}}(|-|, \mathfrak{X}))$$

(where $\text{Hom}_{\mathbf{CTop}}$ is actually the internal Hom of **CTop**, whence we get a controlled set by ‘forgetting’). One should check that this agrees with our explicit construction of $\text{CSing} \circ \text{MaxCtl}$ above and that it generalizes Sing

in the sense that $\text{Sing} = \text{Forget}_\#(\text{CSing} \circ \text{MinCtl})$ (notation as above; see also Remark 4.5 below).

- (v) Define a realization functor $|-|: \mathbf{sCSet} \rightarrow \mathbf{CTop}$ by the coend (in \mathbf{CTop})

$$|X| := \int^n \text{CTop}(X(\mathbf{n})) \otimes |\mathbf{n}|$$

for each simplicial controlled set X (strictly speaking, $|\mathbf{n}|$ should be written as $\text{MinCtl}(|\mathbf{n}|)$; see Remark 4.5 below). This is entirely analogous to the definition (3.1) of the classical realization of a simplicial set, but one should verify that it generalizes the classical realization (in the sense that $\text{Forget}(|\text{MinCtl}_\#(-)|)$ is just the classical realization $|-|: \mathbf{sSet} \rightarrow \mathbf{Top}$).

- (vi) Prove that realization is left adjoint to CSing .

Unfortunately, the second step, constructing \mathbf{CTop} involves nontrivial work in general topology, with many of the same difficulties entailed in finding a ‘convenient category’ of topological spaces (as well as other difficulties). We thus leave this task to the future.

Remark 4.5. By analogy with $\text{CSing} \circ \text{MaxCtl}$, we may also define

$$\text{CSing} \circ \text{MinCtl} := \text{MinCtl}_\#(\text{Sing}): \mathbf{Top} \rightarrow \mathbf{CSet},$$

i.e., for any space \mathfrak{X} , $\text{CSing}(\text{MinCtl}(\mathfrak{X})) := \text{MinCtl} \circ \text{Sing}(\mathfrak{X})$. Here, the notation indicates that to any space \mathfrak{X} can be made into a controlled topological space $\text{MinCtl}(\mathfrak{X})$ by equipping it with its family of compact subspaces.

Proof of Theorem 4.2. Fix \mathfrak{X} and Y . Since we insisted on compatibility with the classical version, the adjunction (3.2) actually determines the isomorphism (4.3) and ensures naturality; it is only a matter of checking that the maps which (3.2) provides are proper or controlled (as appropriate).

Suppose that $f: |Y| \rightarrow \mathfrak{X}$ is proper. We get a natural map $F = (F_n): Y \rightarrow \text{Sing}(\mathfrak{X})$ of simplicial sets, i.e., for each n we get a set map $F_n: Y_n \rightarrow \text{Sing}(\mathfrak{X})_n$. We must check that the F_n are controlled with respect to the control structures of $\text{MaxCtl}(Y_n)$ and $\text{CSing}(\text{MaxCtl}(\mathfrak{X}))$, i.e., that the F_n are proper and have controlled images. Since f is proper, for each compact $K \subseteq \mathfrak{X}$, $f^{-1}(K)$ intersects (the realizations of) only finitely many nondegenerate simplices; thus each compact K intersects the images of only finitely many nondegenerate simplices of $F(Y)$. It follows that in each dimension n , each K intersects the images of only finitely many simplices, so each $F_n(Y_n)$ is controlled. The same kind of argument also shows that each F_n is proper.

Conversely, suppose that $F = (F_n): \text{MaxCtl} \circ Y \rightarrow \text{CSing}(\text{MaxCtl}(\mathfrak{X}))$ is a map of simplicial controlled sets (hence a map of simplicial sets). Naturally, we get a continuous map $f: |Y| \rightarrow \mathfrak{X}$. We must check that f is proper, so fix $K \subseteq \mathfrak{X}$ compact. By properness of the F_n , $f^{-1}(K)$ can intersect the (realizations of) finitely many n -simplices of Y . Then, by local finiteness of Y , $f^{-1}(K)$ can only intersect finitely many nondegenerate simplices. \square

5. HOMOLOGY AND COHOMOLOGY

Let us briefly indicate how we can develop homology and cohomology in our setting. Fix an abelian group G (of coefficients), e.g., $G := \mathbb{Z}$. Given a controlled set X , let $G[X]$ be the set (indeed, abelian group) of formal sums

$$\sum_{x \in X} [x] g_x, g_x \in G,$$

with controlled support, i.e., $\{x \in X : g_x \neq 0\}$ is in $\text{Units}(X)$. We make $G[-]$ into *covariant* functor (from **CSet** to the category **Ab** of abelian groups) in the way suggested by the summation notation: if $f: Y \rightarrow X$ is a controlled map between controlled sets, then

$$f_*(\sum [y] g_y) := \sum [f(y)] g_y$$

(adding coefficients when necessary). Formally, $G[X]$ is just a set of functions $X \rightarrow G$ (or, equivalently, a subgroup of the product $\prod_{x \in X} G$), but this is misleading in terms of functoriality.

Thus, if X is a *simplicial controlled set*, then $G[X]$ (strictly speaking, $G[-] \circ X$) is a simplicial abelian group. We then get a chain complex and homology groups $H_\bullet(G[X])$ in the usual way. We immediately get the following (see, e.g., [2]).

Proposition 5.1. *For any locally compact space \mathfrak{X} , $H_\bullet(G[X])$ is just the (locally finitely supported) Borel–Moore singular homology of \mathfrak{X} .*

Even more trivially, we can recover the usual singular homology.

Proposition 5.2. *For any space \mathfrak{X} , $H_\bullet(G[\text{CSing}(\text{MinCtl}(\mathfrak{X}))])$ (notation as in Remark 4.5) is just the usual (compactly supported) singular homology of \mathfrak{X} .*

To get cohomology, we give a contravariant functor from **CSet** to **Ab** (i.e., a covariant functor $\mathbf{CSet}^{\text{op}} \rightarrow \mathbf{Ab}$). Say that a subset S of a controlled set X is *cocontrolled* if $S \cap T$ is finite for all $T \in \text{Units}(X)$. Let $G[X]^*$ be the set (indeed, abelian group) of functions $X \rightarrow G$ with cocontrolled support. One can then check that this yields a *contravariant* functor (by pulling back functions, as usual).

For any simplicial controlled set X , $G[X]^*$ (strictly speaking, $G[-]^* \circ X$) is a cosimplicial abelian group, and we get a cochain complex and cohomology groups $H^\bullet(G[X]^*)$. One can check the following results for cohomology.

Proposition 5.3. *For any locally compact space \mathfrak{X} , $H^\bullet(G[\text{CSing}(\text{MaxCtl}(\mathfrak{X}))]^*)$ is the usual singular homology of \mathfrak{X} with compact supports.*

Proposition 5.4. *For any space \mathfrak{X} , $H^\bullet(G[\text{CSing}(\text{MinCtl}(\mathfrak{X}))]^*)$ is the usual singular cohomology of \mathfrak{X} .*

Remark 5.5. For any controlled set X and abelian groups G and H , observe that there is an obvious pairing

$$(\text{Hom}_{\mathbf{Ab}}(H, G))[X]^* \times H[X] \rightarrow G,$$

namely the one given by

$$\langle \alpha, \sigma \rangle := \sum_x \alpha(x)(h_x)$$

for $\alpha \in (\text{Hom}(H, G))[X]^*$ (hence α is a function $X \rightarrow \text{Hom}(H, G)$) and $\sigma := \sum [x]h_x \in H[X]$; the sum is actually finite since α has cocontrolled support and σ has controlled support. This evidently induces the usual pairing between homology and cohomology.

Remark 5.6. One may obtain the contravariant functor $G[-]^*$ in a slightly more satisfactory way as follows. First, observe that, for any X , one can give $G[X]$ slightly more structure than just its structure as an abelian group. Given a collection (possibly infinite) $\{\sigma_i\}_{i \in I}$ (I some index set) of elements of $G[X]$, one can form the (possibly infinite) sum $\sum_{i \in I} \sigma_i$ if each $x \in X$ is contained in only finitely many of the supports of the σ_i . That is, $G[X]$ has an additional structure which allows one to take certain infinite sums.

One can axiomatize such objects, namely abelian groups in which certain infinite sums are defined, and obtain a category **CAb** of *controlled abelian groups*. Then one can make **CAb** into a closed category, in particular giving it an internal Hom. Giving G the structure of a controlled abelian group with ‘minimal control’ (i.e., only finite sums are defined), we get a controlled abelian group $\text{Hom}_{\mathbf{CAb}}(\mathbb{Z}[X], G)$ which turns out to be our explicitly-constructed exactly $G[X]^*$.

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